Categorical Hubris, Humility and Atonement

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Structure of the Talk

- **Categorical Hubris:** What do we make of common types like
  - $\forall a.a \to a$, $\forall a.fa \to ga$ and $\forall a.(fa \to a) \to a$
  - Naturality, Dinaturality, Strong Dinaturality ...
- **Categorical Humility:** Wrong answer!!!!! CS got it right ...
  - Reynold's idea was based on *logical relations*
  - Build a relational model over a standard model
- **Categorical Atonement:** What does this mean ....
  - Shift from a categorical universe to a fibrational universe
  - Fibrations, fibred functors and fibred natural transformations

Hubris

- **Reverse:** Wadler's theorems for free for polymorphic functions

$$\text{reverse} :: \forall a.\text{List a} \to \text{List a}$$

satisfying

$$\text{reverse} \ (\text{map} \ f \ xs) = \text{map} \ f \ (\text{reverse} \ xs)$$

- **Categorically:** Ah, naturality! What about types such as

$$\forall a.a \to a \quad \forall a.fa \to ga \quad \forall a.(fa \to a) \to a$$

- **Hubris:** Can we prove properties we want?
  - ... theories such as (strong) dinaturality don't work

Humility

- **Key Idea:** Define judgements $\Gamma \vdash T\ : *$ by
  - Variables: $X_1, ..., X_n \vdash X_i\ : *$
  - Arrow Types: If $\Gamma \vdash T, U\ : *$ then $\Gamma \vdash T \to U\ : *$
  - Forall types: If $\Gamma, X \vdash T\ : *$, then $\Gamma \vdash \forall X.T\ : *$
  - Judgements for defining terms/morphisms: $\Gamma, \Delta \vdash t : T$ where $\Gamma \vdash T\ : *$ and $(x_i : T_i) \in \Delta \Rightarrow \Gamma \vdash T_i\ : *$

- **John Reynolds:** Defined logical relations by defining two semantics of the following form. Let Set be a universe of sets

$$
\begin{aligned}
[\![\Gamma \vdash A : *]\!]_0 &\in \mathsf{Set}^{|\Gamma|} \to \mathsf{Set} \\
[\![\Gamma \vdash A : *]\!]_1 &\in \forall \gamma_1, \gamma_2 \in \mathsf{Set}^{|\Gamma|}.\ \mathsf{Rel}^{|\Gamma|}(\gamma_1, \gamma_2) \to \\
&\qquad \mathsf{Rel}([\![\Gamma \vdash A : *]\!]_0 \gamma_1, [\![\Gamma \vdash A : *]\!]_0 \gamma_2)
\end{aligned}
$$

- **Variables:** If $X_1, ..., X_n \vdash X_i \;: *$, then

$$\begin{array}{rcl} [\![X_1, ..., X_n \vdash X_i \;: *]\!]_0\, \gamma & = & \gamma_i \\ [\![X_1, ..., X_n \vdash X_i \;: *]\!]_1\, r & = & r_i \end{array}$$

- **Arrow Types:** If $\Gamma \vdash T \to U \;: *$, then

$$[\![\Gamma \vdash T \to U \;: *]\!]_0 \gamma = [\![\Gamma \vdash T \;: *]\!]_0\, \gamma \to [\![\Gamma \vdash U \;: *]\!]_0\, \gamma$$

$(f, g) \in [\![\Gamma \vdash T \to U \;: *]\!]_1\, r$ iff

$$(a, b) \in [\![\Gamma \vdash T \;: *]\!]_1\, r \Rightarrow (fa, gb) \in [\![\Gamma \vdash U \;: *]\!]_1\, r$$

- **Intuition:** Parametrically polymorphic functions map related inputs to related outputs.

- **Remark:** This definition makes Rel cartesian closed.

Forall Types: If $\Gamma \vdash \forall X.T\ :*$*, then ...*

- **Forall Types I:** $[\![\Gamma \vdash \forall X.T\ :*]\!]_0\ \gamma$ is

  $$\{f : (S : Set) \to [\![\Gamma, X \vdash T\ :*]\!]_0(S, \gamma)\ \mid \\ R : \mathsf{Rel}(A,B) \Rightarrow (fA, fB) \in [\![\Gamma, X \vdash T\ :*]\!]_1(R, Eq_\gamma)\}$$

  — polymorphic functions satisfying a "uniformity" condition.

- **Forall Types II:** $(f, g) \in [\![\Gamma \vdash \forall X.T\ :*]\!]_1\ r$ iff

  $$R \in \mathsf{Rel}(A,B) \Rightarrow (fA, gB) \in [\![\Gamma, X \vdash T\ :*]\!]_1(R, r)$$

  Two polymorphic functions are related if they map "related types" to "related types"

- **Dependent Types:** In separate (non-fibrational) work, we exploit this idea to define a parametric model of dependent types.

- **Identity Extension Lemma:** If $\Gamma \vdash T\ :*$ then

$$[\![\Gamma \vdash T\ :*]\!]_1(Eq_S) = Eq_{[\![\Gamma \vdash T\ :*]\!]_0 S}$$

  A lemma about types - the relational semantics maps equality relations to equality relations

- **Fundamental Theorem:** First give a standard semantics to terms. If $\Gamma, \Delta \vdash t : T$, then

$$[\![\Gamma, \Delta \vdash t : T]\!]_0 : \forall \gamma : \mathsf{Set}^{|\Gamma|}.[\![\Gamma \vdash \Delta]\!]_0\gamma \to [\![\Gamma \vdash T]\!]_0\gamma$$

  and then prove that if $\gamma_1, \gamma_2 \in \mathsf{Set}^{|\Gamma|}, \rho \in \mathsf{Rel}^{|\Gamma|}(\gamma_1, \gamma_2)$ and also if $\theta_1 \in [\![\Gamma \vdash \Delta]\!]_0\gamma_1$ and $\theta_2 \in [\![\Gamma \vdash \Delta]\!]_0\gamma_2$, then

$$(\theta_1, \theta_2) \in [\![\Gamma \vdash \Delta]\!]_1\rho \Rightarrow ([\![t]\!]_0\ \gamma_1\ \theta_1, [\![t]\!]_0\ \gamma_2\ \theta_2) \in [\![\Gamma \vdash T\ :*]\!]_1\rho$$

  All terms map related inputs to related outputs.

Categorical Atonement

- **Motivation:** A categorical abstraction of a domain of discourse and a logic over it
  - A category $\mathcal{B}$, called the base
  - A category $\mathcal{E}$, called the total category
  - A functor $p : \mathcal{E} \to \mathcal{B}$ mapping each logical formula to the object it is a property of

- **Parametricity:** A logical process for reasoning about programs
  - Properties are typically relatedness
  - But they could be constructive or higher order relations

- **Fibrations:** Define some categories
  - Set is the category of small sets and functions. Rel has as objects binary relations and as morphisms, pairs of functions between the carriers of the relations preserving relatedness. $p : \mathsf{Rel} \to \mathsf{Set} \times \mathsf{Set}$ maps $R : \mathsf{Rel}(X, Y)$ to (X, Y).

- **Semantics of Types:** If $\Gamma \vdash T\ : *$, and $n = |\Gamma|$, then

$$\begin{array}{ccc} |\mathsf{Rel}|^{\mathsf{n}} & \xrightarrow{[\![T]\!]_1} & \mathsf{Rel} \\ \big\downarrow {\scriptstyle |p|^n} & & \big\downarrow {\scriptstyle p} \\ |\mathsf{Set}|^{\mathsf{n}} \times |\mathsf{Set}|^{\mathsf{n}} & \xrightarrow[{[\![T]\!]_0 \times [\![T]\!]_0}]{} & \mathsf{Set} \times \mathsf{Set} \end{array}$$

- **Key Idea:** No action of type semantics on morphisms!!! And can generalise to all fibrations!

Identity Extension Lemma

- **Definition:** Equality defines a functor $Eq : \mathsf{Set} \to \mathsf{Rel}$

- **Identity Extension Lemma:** Simply ...

$$
\begin{array}{ccc}
|\mathsf{Rel}|^{\mathsf{n}} & \xrightarrow{[\![T]\!]_1} & \mathsf{Rel} \\
{\scriptstyle |Eq|^n}\uparrow & & \uparrow{\scriptstyle Eq} \\
|\mathsf{Set}|^{\mathsf{n}} & \xrightarrow[{[\![T]\!]_0}]{} & \mathsf{Set}
\end{array}
$$

- **Why Fibrations:** Equality can be defined in any fibration which is a bifibration with fibred terminal objects, ie logically has Σ-types and truth objects

$$EqX = \Sigma_{\delta_X : X \to X \times X}\ \mathsf{Tr}X$$

- **Recall:** The standard interpretation of a term $\Gamma, \Delta \vdash t : T$ is a function

$$[\![\Gamma, \Delta \vdash t : T]\!]_0 : \forall\gamma : \mathsf{Set}^{|\Gamma|}.[\![\Gamma \vdash \Delta]\!]_0\gamma \to [\![\Gamma \vdash T]\!]_0\gamma$$

or, categorically, as a natural transformation

$$|\mathsf{Set}|^n \overset{[\![\Gamma\vdash\Delta]\!]_0}{\underset{[\![\Gamma\vdash T]\!]_0}{\Downarrow}} \mathsf{Set}$$

- **Question:** But what about the fundamental theorem ... its just a natural transformation $[\![t]\!]_1$ over $[\![t]\!]_0 \times [\![t]\!]_0$

$$|\mathsf{Rel}|^n \overset{[\![\Gamma\vdash\Delta]\!]_1}{\underset{[\![\Gamma\vdash T]\!]_1}{\Downarrow}} \mathsf{Rel}$$

- **Related work:** A Huge literature
  - Most certainly influenced heavily by Dunphy and Reddy
  - Birkedal, Mogelberg, Simpson, Hermida etc are fibrational
  - Exciting non categorical work, eg Bernardy, Dreyer, Neil K
- **Future:** Many directions we hope to travel
  - Dependent Types, Effects and HoTT
  - Constructions on Parametric Models
  - 4-year RA position available on EPSRC grant *Logical Relations for Program Verification*